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BOOTSTRAP PREDICTION INTERVALS IN STATE SPACE MODELS*

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Keywords: Backward representation, Kalman filter, Local Level Model, Unobserved Components.

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Prediction intervals in State Space models can be obtained by assuming Gaussian innovations and using the prediction equations of the Kalman filter, where the true parameters are substituted by consistent estimates. This approach has two limitations. First, it does not incorporate the uncertainty due to parameter estimation. Second, the Gaussianity assumption of future innovations may be inaccurate. To overcome these drawbacks, Wall and Stoffer (2002) propose to obtain prediction intervals by using a bootstrap procedure that requires the backward representation of the model. Obtaining this representation increases the complexity of the procedure and limits its implementation to models for which it exists. The bootstrap procedure proposed by Wall and Stoffer (2002) is further complicated by fact that the intervals are obtained for the prediction errors instead of for the observations. In this paper, we propose a bootstrap procedure for constructing prediction intervals in State Space models that does not need the backward representation of the model and is based on obtaining the intervals directly for the observations. Therefore, its application is much simpler, without loosing the good behavior of bootstrap prediction intervals. We study its finite sample properties and compare them with those of the standard and the Wall and Stoffer (2002) procedures for the Local Level Model. Finally, we illustrate the results by implementing the new procedure to obtain prediction intervals for future values of a real time series.

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1 Introduction

When analyzing economic and financial time series, sometimes it is useful to decompose them into latent components such as, for example, trend, seasonal, cyclic and irregular component which have a direct interpretation; see Harvey (1989) and Durbin and Koopman (2001) for extensive descriptions of Unobserved Component models. The empirical applications of these models are very wide; for instance, the evolution of inflation could be represented by a model with long-run level, seasonal and transitory components; see, for example, Ball et al. (1990), Evans (1991), Kim (1993) and Broto and Ruiz (2006). Cavaglia (1992) analyzes the dynamic behavior of ex-ante real interest differentials across countries by a linear model in which the ex-post real interest differential is expressed as the ex-ante real interest differential (underlying unobserved component) plus the cross country differential inflation forecast error. When modelling financial returns, the volatility can also be modelled as an unobserved component as in the Stochastic Volatility (SV) models proposed by Taylor (1982) and popularized by Harvey et al. (1994).

The parameters of models with unobserved components can be estimated by Quasi-Maximum Likelihood (QML) by casting the model in the State Space (SS) form and using the Kalman filter to obtain the one-step ahead prediction error expression of the Gaussian likelihood. Once the parameters have been estimated, the unknown parameters can be substituted by the corresponding QML estimates, so that the filter provides estimations and predictions of the unobserved components. It also delivers future predictions of the series of interest together with their corresponding mean square errors (MSE). However, these MSEs are based on assuming known parameters and Gaussian errors. Therefore, the corresponding prediction intervals may be inaccurate, because they do not incorporate the variability due to parameter estimation, and also, because the Normal distribution could be different from the true one. In the context of ARIMA models several, authors propose to use bootstrap procedures to construct prediction intervals that overcome these limitations. The seminal paper in this area is Thombs and Schu-

cany (1990) who propose a bootstrap procedure to obtain prediction intervals for $AR(p)$ models based on estimating directly the distribution of the conditional predictions. They propose to incorporate the uncertainty due to parameter estimation by generating bootstrap replicates of the observed series and estimating parameters in each of them. All bootstrap replicates have the same last p values and, consequently, the procedure of Thombs and Schucany (1990) requires the use of the backward representation of the model. The need of this representation complicates computationally the procedure and limits its implementation to models with it. On the other hand, Pascual et al. (2004) show that when trying to incorporate parameter uncertainty in prediction intervals, there is not need of fixing the last p observations of each bootstrap replicate. They only fix the last p observations to obtain bootstrap replicates of future values of the series but the estimated parameters are bootstrapped without fixing any observation in the sample. Consequently, the backward representation is unnecessary, which simplifies the construction of bootstrap prediction intervals and allows to extend the procedure to models without such representation.

Unlike ARIMA models, models with unobserved components may have several disturbances. Therefore, the bootstrap procedures proposed by Thombs and Schucany (1990) and Pascual et al. (2004) cannot be directly applied to them. However, the innovation form of SS models has only one disturbance. Consequently, Wall and Stoffer (2002) propose using it to obtain prediction intervals for future observations. However, as in Thombs and Schucany (1990), the bootstrap procedure proposed by Wall and Stoffer (2002) requires the use of the backward representation. Furthermore, its implementation is complicated by the fact that the bootstrap density of the prediction errors is obtained in two steps. First, the density that takes into account the parameter estimation uncertainty is obtained and then the density that takes into in account the variability of future innovations. Finally, these two densities are combined in the overall density of the prediction errors that is itself used to obtain the density of future observations. They show that their procedure works well in the context of Gaussian SS models. Moreover, Pfeffermann and Tiller (2005) show that the bootstrap estimator of the

underlying unobserved component based on the innovation form is asymptotically consistent. However, it is computationally complicated to implement in practice and to extend to more general models the bootstrap procedure proposed by Wall and Stoffer (2002). Alternatively, following Pascual et al. (2004), in this paper we propose a bootstrap procedure to obtain directly prediction intervals of future observations in SS models that does not require the backward representation. As in Wall and Stoffer (2002), our proposed bootstrap procedure is based on the innovation form of SS models. We show that the new procedure has the advantage of being much simpler without loosing the good behavior of bootstrap prediction intervals. The finite sample behavior of the new intervals is compared with intervals based on the standard Kalman filter and on the Wall and Stoffer (2002) procedure in the context of Gaussian and non-Gaussian linear SS models.

The rest of the paper is organized as follows. Section 2 describes the Kalman filter, the innovation representation and the construction of prediction intervals. Section 3 deals with the construction of bootstrap prediction intervals in SS models. We first describe the procedure proposed by Wall and Stoffer (2002) (WS) and then the new procedure proposed in this paper. Section 4 analyzes the finite sample properties of the new procedure by means of Monte Carlo experiments. They are then compared with those of the standard and WS prediction intervals. Section 5 presents an application of the new bootstrap procedure to a real time series. Section 6 concludes the paper with our conclusions and some suggestions for future research.

2 State Space Models and the Kalman Filter

Consider the following SS model,

$$y_t = \boldsymbol{Z}_t\alpha_t + \boldsymbol{d}_t + \varepsilon_t, \tag{1a}$$

$$\alpha_t = \boldsymbol{T}_t\alpha_{t-1} + \boldsymbol{c}_t + \boldsymbol{R}_t\eta_t, \quad t = 1, \ldots, T. \tag{1b}$$

where y_t is a univariate time series observed at time t, $\boldsymbol{Z}_t$ is a $1 \times m$ vector, $\boldsymbol{d}_t$ is a scalar and ε_t is a serially uncorrelated disturbance with zero mean and variance

$\boldsymbol{H}_t$. On the other hand, α_t is the $m \times 1$ vector of unobservable state variables, $\boldsymbol{T}_t$ is an $m \times m$ matrix, $\boldsymbol{c}_t$ is an $m \times 1$ vector, $\boldsymbol{R}_t$ is an $m \times g$ matrix and η_t is a $g \times 1$ vector of serially uncorrelated disturbances with zero mean and covariance matrix $\boldsymbol{Q}_t$. Finally, the disturbances ε_t and η_t are uncorrelated with each other in all time periods. The system matrices $\{\boldsymbol{Z}_t, \boldsymbol{T}_t, \boldsymbol{Q}_t, \boldsymbol{H}_t, \boldsymbol{R}_t, \boldsymbol{c}_t, \boldsymbol{d}_t\}$ are assumed to be *time-invariant* and the subindex t is dropped from them. The specification of the SS system is completed with the initial state vector, α_0, which has mean $\mathbf{a}_0$ and covariance matrix P_0.

The Kalman filter is a recursive algorithm for estimating the state vector, α_t, and its MSE based on the information available at time t. These estimates are given by the following updating equations

$$\mathbf{a}_t = \mathbf{a}_{t|t-1} + P_{t|t-1}\boldsymbol{Z}'F_t^{-1}v_t, \tag{2a}$$

$$P_t = P_{t|t-1} - P_{t|t-1}\boldsymbol{Z}'F_t^{-1}\boldsymbol{Z}P_{t|t-1}, \tag{2b}$$

where $\mathbf{a}_{t|t-1}$ and $P_{t|t-1}$ are the one-step ahead prediction of the state and its MSE which given by the following prediction equations

$$\mathbf{a}_{t|t-1} = \boldsymbol{T}\mathbf{a}_{t-1} + \mathbf{c} \tag{2c}$$

$$P_{t|t-1} = \boldsymbol{T}P_{t-1}\boldsymbol{T}' + \boldsymbol{R}\boldsymbol{Q}\boldsymbol{R}'. \tag{2d}$$

Finally, $v_t = y_t - \boldsymbol{d} - \boldsymbol{Z}\mathbf{a}_{t|t-1}$ is the innovation and F_t is its variance given by $F_t = \boldsymbol{Z}P_{t|t-1}\boldsymbol{Z'} + \boldsymbol{H}$. When the model in (1) is time-invariant the Kalman filter converges to a steady state with covariance matrices $P_{t|t-1} = \overline{P}$ and $P_t = a\overline{P}$, where a is a constant, and $F_t = \overline{F}$; see Anderson and Moore (1979) and Harvey (1989).

If the model is conditionally Gaussian, then $\mathbf{a}_t = \underset{t}{\mathbb{E}}[\alpha_t]$, where the t under the expectation means that it is conditional on the information available at time t, and has minimum MSE. However, if the conditional Gaussianity assumption is not fulfilled, the filter provides estimates with minimum MSE among the linear estimators. Finally, if the initial conditions are not given as a part of the model specification, then, it is possible to initialize the filter via one of the following

approaches. First, when the state is generated by a stationary process, the filter can be initialized by the marginal mean and variance of the state. When the state is non stationary, we can assume that α_0 has a diffuse distribution with zero mean and covariance matrix $P_0 = kI_m$, where $k \longrightarrow \infty$, which is equivalent to using the first observations of the series as initial values. Also, in the non stationary case, it is also possible to assume that the initial state, α_0, is fixed, i.e. its distribution is degenerated with $P_0 = 0$. In this case, its elements must be estimated by treating them as unknown parameters in the model; see, for instance, Harvey (1989) for more details.

Although, the SS model in (1) has several disturbances, it is possible to express it in what is known as the Innovation Form (IF) which has a unique disturbance

$$y_t = \boldsymbol{Z}\mathbf{a}_{t|t-1} + \boldsymbol{d} + v_t. \tag{3a}$$

Combining equations (2c) and (2a) it is straightforward to see that

$$\mathbf{a}_{t+1|t} = \boldsymbol{T}\mathbf{a}_{t|t-1} + \mathbf{c} + \overline{\boldsymbol{K}F}^{-1} v_t, \tag{3b}$$

where $\overline{\boldsymbol{K}} = \boldsymbol{T}\overline{P}\boldsymbol{Z}'$. Equations (3a) and (3b) conform the IF.

As an illustration, consider the Local Level model given by

$$y_t = \mu_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{IID}(0, \sigma_\varepsilon^2), \tag{4a}$$

$$\mu_t = \mu_{t-1} + \eta_t, \qquad \eta_t \sim \mathcal{IID}(0, q\sigma_\varepsilon^2), \tag{4b}$$

where the unobserved state, α_t, is the level of the series, denoted by μ_t, that evolves over time following a random walk. In this model $\boldsymbol{Z} = \boldsymbol{T} = \boldsymbol{R} = 1$, $\boldsymbol{c} = \boldsymbol{d} = 0$, $\boldsymbol{H} = \sigma_\varepsilon^2$ and $\boldsymbol{Q} = q\sigma_\varepsilon^2$, where q is known as the signal to noise ratio. The corresponding IF is given by

$$y_t = m_{t|t-1} + v_t, \tag{5a}$$

$$m_{t+1|t} = m_{t|t-1} + \left(\frac{\overline{P}}{\overline{F}}\right) v_t, \tag{5b}$$

where $\overline{F} = \overline{P} + \sigma_\varepsilon^2$. Finally, if the initial conditions are assumed to be given by

a diffuse distribution, then the filter can be initialized using the first observation, i.e. $m_{1|0} = y_1$ and $P_{1|0} = \sigma_\varepsilon^2$.

After the last observation is available, the Kalman filter can still be run without the updating equations, in (2a) and (2b). In this case, the k-step ahead predictions of the underlying unobserved components are given by

$$\mathbf{a}_{T+k|T} = \boldsymbol{T}^k \mathbf{a}_T + \sum_{j=0}^{k-1} \boldsymbol{T}^j \mathbf{c}, \qquad k = 1, 2, \ldots, \tag{6a}$$

while the associated MSE matrix is given by

$$P_{T+k|T} = \left(\boldsymbol{T}^k\right) P_T \left(\boldsymbol{T}^k\right)' + \sum_{j=0}^{k-1} \left[\left(T^j\right) \boldsymbol{R}\boldsymbol{Q}\boldsymbol{R}' \left(\boldsymbol{T}^j\right)'\right], k = 1, 2, \ldots, \tag{6b}$$

where $P_T = \overline{P}$. The k-step ahead prediction of y_{T+k} is given by

$$\widetilde{y}_{T+k|T} = \boldsymbol{Z}\mathbf{a}_{T+k|T} + \boldsymbol{d}, k = 1, 2, \ldots, \tag{7a}$$

with prediction MSE given by

$$MSE(\widetilde{y}_{T+k|T}) \equiv F_{T+k|T} = \boldsymbol{Z} P_{T+k|T} \boldsymbol{Z'} + \boldsymbol{H}, k = 1, 2, \ldots. \tag{7b}$$

Consequently, assuming that future prediction errors are Normally distributed, prediction intervals for y_{T+k} are given by

$$\left[\widetilde{y}_{T+k|T} - z_{1-\alpha/2}\sqrt{F_{T+k|T}},\ \widetilde{y}_{T+k|T} + z_{1-\alpha/2}\sqrt{F_{T+k|T}}\right], \tag{8}$$

where $z_{1-\alpha/2}$ is the $\left(1 - \frac{\alpha}{2}\right)$-percentile of the Standard Normal distribution; see, for example, Durbin and Koopman (2001).

The point prediction $\widetilde{y}_{T+k|T}$ and its MSE in (7a) and (7b) respectively, are obtained assuming known parameters. However, in practice, the unknown parameters are substituted by consistent estimates. In this paper, we consider the QML estimator due to its well known asymptotic properties; see, for example, Harvey (1989) and Durbin and Koopman (2001). Hence, denoting by $\hat{\boldsymbol{Z}}, \hat{\boldsymbol{d}}$ and $\hat{\boldsymbol{H}}$ the system of matrices where the parameters have been substituted by their

QML estimates, the k-step ahead prediction of y_{T+k} is given by

$$\hat{y}_{T+k|T} = \hat{\boldsymbol{Z}}\hat{\mathbf{a}}_{T+k|T} + \hat{\boldsymbol{d}} \tag{9a}$$

with estimated MSE given by

$$\hat{MSE}\left(\hat{y}_{T+k|T}\right) = \hat{F}_{T+k|T} = \hat{\boldsymbol{Z}}\hat{\overline{P}}\hat{\boldsymbol{Z}}' + \hat{\boldsymbol{H}} \tag{9b}$$

where $\hat{\mathbf{a}}_{T+k|T}, \hat{\overline{P}}$ given by the filter run with QML estimates. Consequently, in practice, the prediction intervals for future values of y_t are given by

$$\left[\hat{y}_{T+k|T} - z_{1-\alpha/2}\sqrt{\hat{F}_{T+k|T}},\ \hat{y}_{T+k|T} + z_{1-\alpha/2}\sqrt{\hat{F}_{T+k|T}}\right]. \tag{10}$$

We call the interval in (10) as standard (ST).

Note that the MSE in (9b) does not take into in account the uncertainty due to parameter estimation and therefore, the corresponding prediction intervals, in (10), underestimate, in general, the variability of the forecasting error. Moreover, these intervals could have inaccurate coverage when the prediction errors are not Gaussian.

Consider again the local level model. In this case, the estimated predictions of future observations, y_{T+k}, are given by

$$\hat{y}_{T+k|T} = \hat{m}_T, k = 1, 2, \ldots, \tag{11a}$$

with MSE

$$\hat{F}_{T+k|T} = \hat{\overline{P}} + k\hat{q}\hat{\sigma}_\varepsilon^2 + \hat{\sigma}_\varepsilon^2. \tag{11b}$$

Finally, the ST prediction interval for y_{T+k} is

$$\left[\hat{m}_T - z_{1-\alpha/2}\sqrt{\hat{\overline{P}} + \hat{\sigma}_\varepsilon^2\left(1 + k\hat{q}\right)},\ \hat{m}_T + z_{1-\alpha/2}\sqrt{\hat{\overline{P}} + \hat{\sigma}_\varepsilon^2\left(1 + k\hat{q}\right)}\right]. \tag{12}$$

3 Bootstrap Prediction Intervals in State Space Models

In this section we describe the bootstrap procedure proposed by Wall and Stoffer (2002) for constructing prediction intervals in SS models. Then, we propose a new simpler procedure which avoids using the backward representation and obtains directly the intervals of future observations.

3.1 The Wall Stoffer Procedure

Wall and Stoffer (2002) propose to use bootstrap procedures to construct prediction intervals for future values of series modeled by linear SS models. Their procedure is based on the IF in (3) that only has one disturbance. Following Thombs and Schucany (1990), they propose to use the backward SS representation to generate bootstrap replicates of the series with fixed last observations. These replicates are used to incorporate in the density of the prediction errors, the uncertainty due to parameter estimation. Then, they obtain the density of the prediction errors constructed when considering that the parameters are fixed. Finally, combining both densities, they obtain the density of the conditional forecast errors and use it for constructing the corresponding bootstrap prediction interval. Next, we describe in detail the Wall and Stoffer (2002) procedure.

The backward representation of SS models is based on the IF in (3). To simplify the procedure, we consider that $\boldsymbol{d} = \boldsymbol{c} = 0$. Let's define $v_t^s = \frac{v_t}{\sqrt{\overline{F}}}$, $t = 1, \ldots, T$, the standardized innovations. The following equations represent the backward recursion of the SS model in (1)

$$y_t = \boldsymbol{N}_t \tau_{t+1} - \boldsymbol{L}_t \mathbf{a}_{t|t-1} + \boldsymbol{M}_t v_t^s, \quad t = T-1, \ldots, 1, \tag{13a}$$

$$\tau_t = \boldsymbol{T}' \tau_{t+1} + \boldsymbol{A}_t \mathbf{a}_{t|t-1} - \boldsymbol{B}_t v_t^s, \quad t = T-1, \ldots, 1, \tag{13b}$$

where τ_t is the reverse time estimate of the state vector with $\tau_T = \boldsymbol{V}_T^{-1} \mathbf{a}_{T|T-1}$. The matrices in the backward recursions are given by $\boldsymbol{N}_t = \boldsymbol{Z}\,\boldsymbol{V}_t\,\boldsymbol{T}' + \overline{F\,\boldsymbol{K}}'$, $\boldsymbol{L}_t = \overline{F}^{1/2}\boldsymbol{B}_t' - \boldsymbol{Z}\,\boldsymbol{V}_t\boldsymbol{A}_t$, $\boldsymbol{M}_t = \overline{F}^{1/2}\boldsymbol{C}_t - \boldsymbol{Z}\,\boldsymbol{V}_t\boldsymbol{B}_t$, $\boldsymbol{A}_t = \boldsymbol{V}_t^{-1} - \boldsymbol{T}'\,\boldsymbol{V}_{t+1}^{-1}\,\boldsymbol{T}$, $\boldsymbol{B}_t =$

$\boldsymbol{T}'\boldsymbol{V}_{t+1}^{-1}\overline{\boldsymbol{K}}$
$\overline{F}^{1/2}$, $\boldsymbol{C}_t = \boldsymbol{I} - \overline{F}^{1/2}\overline{\boldsymbol{K}}'\boldsymbol{V}_{t+1}^{-1}\overline{\boldsymbol{K}F}^{1/2}$, and $\boldsymbol{V}_{t+1} = \boldsymbol{T}\boldsymbol{V}_t\boldsymbol{T}' + \overline{\boldsymbol{K}F\boldsymbol{K}}'$. These matrices are computed together with the forward Kalman filter with $\boldsymbol{V}_1 = \mathbb{E}\left[\mathbf{a}_{1|0}\mathbf{a}_{1|0}'\right]$.

Consider again the local level model. Its backward representation is given by

$$y_t = N_t r_{t+1} - L_t m_{t|t-1} + M_t v_t^s, \quad t = T-1, \ldots, 1, \tag{14a}$$

$$r_t = r_{t+1} + A_t m_{t|t-1} - B_t v_t^s, \quad t = T-1, \ldots, 1, \tag{14b}$$

where $\tau_T = V_T^{-1} m_{T|T-1}, N_t = V_t + \overline{FP}$, $L_t = \overline{F}^{1/2}B_t - V_t A_t$, $M_t = \overline{F}^{1/2}C_t - V_t B_t$, $A_t = V_t^{-1} - V_{t+1}^{-1}$, $B_t = V_{t+1}^{-1}\overline{PF}^{1/2}$, $C_t = 1 - \overline{FP}^2 V_{t+1}^{-1}, V_{t+1} = V_t + \overline{FP}^2$, and $V_1 = \overline{P}$.

Notice that, as explained before, in practice the parameters are unknown and, consequently, the backward recursion in (13) should be carried out by substituting the unknown parameters by the corresponding QML estimates. In this case, the backward estimates of the state are denoted by $\hat{\tau}_t$ for $t = 1, \ldots, T$.

The WS algorithm to obtain bootstrap prediction intervals of y_{T+k} consists on the following steps:

Step 1: Estimate the parameters of model (1) by QML, $\hat{\theta}$, and construct the standardized innovations $\{\hat{v}_t^s; 1 \leq t \leq T\}$.

Step 2: Construct a sequence of bootstrap standardized innovations $\{\hat{v}_t^{s*}; 1 \leq t \leq T+K\}$ via random draws with replacement from the standardized innovations, $\hat{v}_t^s$, with $\hat{v}_T^{*s} = \hat{v}_T^s$.

Step 3: Construct a bootstrap replicate of the series, $\{y_t^*; 1 \leq t \leq T-1\}$ via the backward SS model, in (13), with estimated parameters, $\theta = \hat{\theta}$, using the innovations $\{\hat{v}_t^{s*}; 1 \leq t \leq T-1\}$ and keeping $y_T^* = y_T$ fixed. Estimate the parameters of the model in order to obtain a bootstrap replicate, $\hat{\theta}^*$, of them.

Step 4: Generate conditional forecasts $\left\{y_{T+k|T}^*; 1 \leq k \leq K\right\}$ via the IF estimated

parameters and bootstrap errors

$$\mathbf{a}^*_{T+k|T} = \hat{\boldsymbol{T}}^k \hat{\mathbf{a}}_{T|T-1} + \sum_{j=0}^{k-1} \hat{\mathbf{T}}^{k-1-j} \hat{\overline{\mathbf{K}}} \hat{\overline{F}}^{-1} \hat{v}^*_{T+j}, \tag{15a}$$

$$y^*_{T+k|T} = \hat{\boldsymbol{Z}}\, \hat{\boldsymbol{T}}^k \hat{\mathbf{a}}_{T|T-1} + \hat{\boldsymbol{Z}} \sum_{j=0}^{k-1} \hat{\mathbf{T}}^{k-1-j} \hat{\overline{\mathbf{K}}} \hat{\overline{F}}^{-1} \hat{v}^*_{T+j} + \hat{v}^*_{T+k}, k = 1, \ldots, . \tag{15b}$$

Step 5: Construct the conditional forecast values $\left\{ \hat{y}^*_{T+k|T}; 1 \leq k \leq K \right\}$ via the IF with bootstrap parameters and future errors equal to zero, i.e.

$$\hat{\mathbf{a}}^*_{T+k|T} = \hat{\boldsymbol{T}}^{*k} \hat{\mathbf{a}}_{T|T-1} \tag{16a}$$

$$\hat{y}^*_{T+k|T} = \hat{\boldsymbol{Z}}^* \hat{\boldsymbol{T}}^{*k} \hat{\mathbf{a}}_{T|T-1},\ k = 1, \ldots, . \tag{16b}$$

where $\hat{\mathbf{a}}^*_{T|T-1} = \hat{\mathbf{a}}_{T|T-1}$

Step 6: Finally, compute the bootstrap forecast error by

$$d^*_k = y^*_{T+k|T} - \hat{y}^*_{T+k|T},\ \text{for } k = 1, 2, \ldots, K\ .$$

Steps 2 to 6 are repeated B times.

Notice that this procedure does not approximate directly the conditional distribution of y_{T+k} but the distribution of the prediction errors. In step 4 the bootstrap replicates $y^*_{T+k|T}$ are constructed using the estimated parameters. They incorporate the uncertainty due to the fact that when predicting, future innovations are equal to zero while in fact they are not. However these bootstrap replicates do not incorporate the uncertainty due to parameter estimation. Then, in step 5 the bootstrap replicates $\hat{y}^*_{T+k|T}$ incorporate the variability attributable to parameter estimation through the use of $\hat{\theta}^*$ instead of $\hat{\theta}$. However, in $\hat{y}^*_{T+k|T}$, future innovations are assumed to be zero. Finally, the conditional bootstrap prediction errors, d^*_k, are computed as the difference between $y^*_{T+k|T} - \hat{y}^*_{T+k|T}$. The corresponding prediction intervals, denoted by WS, are centered at the point prediction $\widetilde{y}_{T+k}$.

They are given by

$$\left[\widetilde{y}_{T+k|T} + \boldsymbol{Q}^*_{\alpha/2,d_k^*},\ \widetilde{y}_{T+k|T} + \boldsymbol{Q}^*_{1-\alpha/2,d_k^*}\right] \tag{17}$$

where $\boldsymbol{Q}^*_{\alpha/2,d_k^*}$ is the $\frac{\alpha}{2}$-percentile of the empirical conditional bootstrap distribution of the k-step ahead prediction errors of y_{T+k}.

3.2 A New Bootstrap Procedure

Our proposal is to construct bootstrap prediction intervals approximating the conditional distribution of y_{T+k} by the distribution of bootstrap replicates that incorporate simultaneously the variability due to parameter estimation and the uncertainty due to unknown future innovations without using the backward filter.

The proposed procedure consists on the following steps:

Step 1: Estimate the parameters of model (1) by QML, $\hat{\theta}$, and obtain the standardized innovations $\{\hat{v}_t^s; 1 \leq t \leq T\}$.

Step 2: Construct a sequence of bootstrap standardized innovations $\{\hat{v}_t^{s*}; 1 \leq t \leq T+K\}$ via random draws with replacement from the standardized innovations, $\hat{v}_t^s$.

Step 3: Compute a bootstrap replicate $\{\hat{y}_t^*; 1 \leq t \leq T\}$ by means of the IF in (3) using $\hat{v}_t^{s*}$ and the estimated parameters, $\hat{\theta}$. Estimate the corresponding bootstrap parameters, $\hat{\theta}^*$. Next, run the Kalman filter with $\hat{\theta}^*$ in order to obtain bootstrap replicates of the state vector at time T which incorporate the uncertainty due to parameter estimation, $\hat{\mathbf{a}}^*_{T|T-1}$.

Step 4: Obtain the conditional bootstrap predictions $\left\{\hat{y}^*_{T+k|T}; 1 \leq k \leq K\right\}$ by the

following expressions

$$\begin{aligned}\hat{\mathbf{a}}^*_{T+k|T} &= \hat{\boldsymbol{T}}^{*k}\hat{\mathbf{a}}^*_{T|T-1} + \sum_{j=0}^{k-1}\hat{\mathbf{T}}^{*k-1-j}\hat{\overline{\mathbf{K}}}^*\hat{\overline{F}}^{*-1}\hat{v}^*_{T+j},\\ \hat{y}^*_{T+k|T} &= \hat{\boldsymbol{Z}}^*\hat{\boldsymbol{T}}^{*k}\hat{\mathbf{a}}^*_{T|T-1}\\ &+ \hat{\boldsymbol{Z}}^*\sum_{j=0}^{k-1}\hat{\mathbf{T}}^{*k-1-j}\hat{\overline{\mathbf{K}}}^*\hat{\overline{F}}^{*-1}\hat{v}^*_{T+j} + \hat{v}^*_{T+k}, k = 1,\ldots,\end{aligned}$$

where $\hat{y}^*_T = y_T$.

Steps 2 to 4 are repeated B times.

The empirical distribution of $\hat{y}^*_{T+k|T}$ incorporates both the variability due to unknown future innovations and the variability due to parameter estimation in just one step. The procedure above, denoted as State Space Bootstrap (SSB), has three advantages over the WS procedure. First, it does not require to use the backward representation. Second, it is simpler as a unique set of bootstrap replicates of future observations is required instead of two as in the WS procedure. Third, unlike the WS procedure, in step 5, we do not fix $\hat{\mathbf{a}}^*_{T|T-1} = \hat{\mathbf{a}}_{T|T-1}$ because this value depends on the estimated parameters, and therefore it should be allowed to vary among bootstrap replicates in order to incorporate the uncertainty due to parameter estimation.

Finally, bootstrap prediction intervals are constructed directly by the percentile method[1]. Hence, bootstrap prediction intervals are given by

$$\left[\boldsymbol{Q}^*_{\alpha/2,\hat{y}^*_{T+k|T}},\ \boldsymbol{Q}^*_{1-\alpha/2,\hat{y}^*_{T+k|T}}\right] \tag{18}$$

where $\boldsymbol{Q}^*_{\alpha/2,\hat{y}^*_{T+k|T}}$ is the $\frac{\alpha}{2}$-percentile of the empirical bootstrap distribution of the k-step ahead prediction of y_{T+k}.

4 Finite Sample Properties

In this section, we analyze the finite sample properties of the SSB prediction intervals and compare them with those of the ST and WS intervals when the series are generated by the local level model in (4).

Simulation results are based on $R = 1000$ replicates of series of sizes $T = 50, 100$ and 500. The parameters of the model have been chosen to cover a wide range of different situations from cases in which the noise is large relative to the signal, i.e. q is small, to cases in which q is large. In particular, we consider $q = \{0.1, 1, 2\}$. With respect to the disturbances, we consider two distributions, Gaussian and a centered and re-scaled Chi-square with 1 degree of freedom[2], $\chi^2_{(1)}$. For each simulated series, $\{y_1^r, \ldots, y_T^r\}$, $r = 1, 2, \ldots, R$, we first generate $B = 1000$ observations of y_{T+k}^r for prediction horizons $k = 1, 5$ and 15, and then obtain, 95% prediction intervals computed using, the ST intervals in (12), the WS intervals in (17) and the SSB intervals in (18). Finally, we compute the coverage of each of these intervals as well as the length and the percentage of observations left out on the right size and on the left size of the limits of the prediction intervals[3].

Table 1: Monte Carlo Average coverages, length and percentage of observations left out on the right and on the left of the prediction intervals constructed using ST, WS and SSB when ε_t is $\mathcal{N}(0,1)$, η_t is $\mathcal{N}(0,q)$ and the nominal coverage is 95%

	k	Mean coverage			Mean coverage in tails			Mean length		
		ST	WS	SSB	ST Below / Above	WS Below / Above	SSB Below / Above	ST	WS	SSB
					$T = 50$					
	1	0.927	0.935	0.936	0.036 / 0.037	0.030 / 0.035	0.031 / 0.033	4.530	4.597	4.774
$q = 0.1$	5	0.927	0.940	0.943	0.036 / 0.037	0.029 / 0.031	0.028 / 0.029	5.182	5.285	5.539
	15	0.915	0.928	0.940	0.042 / 0.042	0.035 / 0.037	0.030 / 0.031	6.460	6.633	7.052
	1	0.936	0.923	0.928	0.029 / 0.035	0.036 / 0.041	0.036 / 0.035	6.157	6.250	6.280
$q = 1$	5	0.927	0.921	0.938	0.035 / 0.039	0.037 / 0.042	0.032 / 0.031	9.722	9.718	10.274
	15	0.914	0.909	0.934	0.041 / 0.045	0.043 / 0.047	0.033 / 0.033	15.258	15.194	16.469
	1	0.938	0.930	0.930	0.032 / 0.029	0.036 / 0.034	0.036 / 0.034	7.424	7.56	7.433
$q = 2$	5	0.926	0.924	0.931	0.037 / 0.036	0.038 / 0.038	0.034 / 0.034	12.849	12.880	13.088
	15	0.918	0.915	0.930	0.041 / 0.041	0.042 / 0.042	0.035 / 0.035	20.889	20.830	21.632
					$T = 100$					
	1	0.945	0.941	0.943	0.025 / 0.030	0.031 / 0.028	0.026 / 0.031	4.569	4.576	4.618
$q = 0.1$	5	0.945	0.942	0.948	0.025 / 0.030	0.030 / 0.028	0.024 / 0.029	5.206	5.238	5.334
	15	0.938	0.938	0.945	0.029 / 0.033	0.032 / 0.030	0.026 / 0.030	6.498	6.575	6.743
	1	0.944	0.940	0.939	0.028 / 0.028	0.030 / 0.029	0.030 / 0.031	6.271	6.314	6.278
$q = 1$	5	0.939	0.937	0.942	0.031 / 0.030	0.032 / 0.031	0.029 / 0.029	9.874	9.873	10.120
	15	0.934	0.932	0.940	0.033 / 0.033	0.034 / 0.034	0.030 / 0.030	15.547	15.521	16.165
	1	0.945	0.937	0.939	0.028 / 0.027	0.032 / 0.030	0.031 / 0.030	7.476	7.537	7.460
$q = 2$	5	0.939	0.938	0.939	0.030 / 0.030	0.031 / 0.031	0.031 / 0.031	13.137	13.155	13.210
	15	0.935	0.935	0.937	0.032 / 0.032	0.032 / 0.033	0.031 / 0.031	21.509	21.539	21.758
					$T = 500$					
	1	0.946	0.948	0.945	0.027 / 0.027	0.025 / 0.028	0.028 / 0.027	4.592	4.577	4.582
$q = 0.1$	5	0.946	0.947	0.946	0.026 / 0.028	0.025 / 0.028	0.027 / 0.027	5.217	5.206	5.223
	15	0.946	0.945	0.945	0.026 / 0.029	0.026 / 0.029	0.027 / 0.028	6.515	6.477	6.511
	1	0.948	0.948	0.947	0.029 / 0.023	0.027 / 0.025	0.027 / 0.025	6.339	6.335	6.314
$q = 1$	5	0.948	0.947	0.947	0.027 / 0.025	0.027 / 0.026	0.028 / 0.025	10.075	10.049	10.073
	15	0.947	0.946	0.947	0.027 / 0.026	0.027 / 0.027	0.027 / 0.026	15.956	15.919	15.944
	1	0.947	0.945	0.947	0.027 / 0.026	0.029 / 0.026	0.027 / 0.026	7.563	7.546	7.540
$q = 2$	5	0.948	0.948	0.948	0.027 / 0.027	0.027 / 0.025	0.027 / 0.027	13.418	13.446	13.387
	15	0.947	0.948	0.947	0.027 / 0.026	0.026 / 0.026	0.026 / 0.027	22.066	22.112	22.051

Table 2: Monte Carlo Average coverages, length and percentage of observations left out on the right and on the left of the prediction intervals constructed using ST, WS and SSB when ε_t is $\chi^2_{(1)}$, η_t is $\mathcal{N}(0, q)$ and the nominal coverage is 95%

Case	k	Mean coverage			Mean coverage in tails			Mean length		
		ST	WS	SSB	ST Below / Above	WS Below / Above	SSB Below / Above	ST	WS	SSB
$T = 50$										
$q = 0.1$	1	0.941	0.940	0.942	0.010 / 0.049	0.030 / 0.030	0.027 / 0.031	4.513	4.909	4.734
	5	0.943	0.934	0.946	0.013 / 0.044	0.039 / 0.027	0.027 / 0.026	5.221	5.507	5.596
	15	0.930	0.919	0.950	0.025 / 0.045	0.053 / 0.029	0.027 / 0.023	6.572	6.665	7.329
$q = 1$	1	0.935	0.932	0.934	0.026 / 0.039	0.034 / 0.034	0.034 / 0.032	6.200	6.514	6.459
	5	0.926	0.926	0.930	0.034 / 0.040	0.040 / 0.034	0.038 / 0.032	9.682	9.803	9.919
	15	0.913	0.914	0.923	0.042 / 0.045	0.041 / 0.036	0.045 / 0.041	15.126	15.176	15.597
$q = 2$	1	0.937	0.933	0.932	0.028 / 0.035	0.034 / 0.034	0.035 / 0.033	7.378	7.714	7.575
	5	0.927	0.924	0.927	0.035 / 0.038	0.040 / 0.036	0.038 / 0.035	12.805	12.897	12.957
	15	0.919	0.917	0.923	0.040 / 0.041	0.043 / 0.040	0.040 / 0.037	20.839	20.880	21.236
$T = 100$										
$q = 0.1$	1	0.947	0.939	0.943	0.006 / 0.048	0.033 / 0.028	0.027 / 0.029	4.552	4.773	4.710
	5	0.946	0.937	0.942	0.010 / 0.043	0.037 / 0.026	0.031 / 0.027	5.196	5.356	5.414
	15	0.939	0.929	0.944	0.021 / 0.040	0.045 / 0.026	0.032 / 0.024	6.491	6.597	6.912
$q = 1$	1	0.942	0.939	0.943	0.021 / 0.037	0.030 / 0.030	0.030 / 0.027	6.244	6.483	6.501
	5	0.937	0.936	0.939	0.028 / 0.034	0.035 / 0.029	0.022 / 0.035	9.813	9.919	10.017
	15	0.932	0.930	0.935	0.033 / 0.035	0.037 / 0.033	0.028 / 0.032	15.438	15.445	15.742
$q = 2$	1	0.947	0.944	0.945	0.022 / 0.031	0.028 / 0.028	0.027 / 0.028	7.507	7.685	7.686
	5	0.942	0.941	0.943	0.028 / 0.030	0.031 / 0.028	0.030 / 0.027	13.220	13.307	13.399
	15	0.938	0.937	0.940	0.030 / 0.031	0.033 / 0.030	0.032 / 0.028	21.659	21.752	21.941
$T = 500$										
$q = 0.1$	1	0.948	0.940	0.950	0.006 / 0.045	0.033 / 0.026	0.023 / 0.027	4.575	4.697	4.707
	5	0.948	0.939	0.947	0.011 / 0.041	0.036 / 0.025	0.028 / 0.025	5.184	5.272	5.314
	15	0.946	0.937	0.946	0.019 / 0.035	0.039 / 0.024	0.031 / 0.024	6.455	6.507	6.631
$q = 1$	1	0.947	0.947	0.948	0.020 / 0.033	0.020 / 0.033	0.027 / 0.026	6.338	6.492	6.472
	5	0.948	0.948	0.948	0.024 / 0.028	0.029 / 0.023	0.027 / 0.025	10.073	10.181	10.137
	15	0.947	0.946	0.947	0.025 / 0.027	0.029 / 0.024	0.028 / 0.025	15.952	15.983	15.957
$q = 2$	1	0.944	0.945	0.944	0.026 / 0.030	0.027 / 0.028	0.029 / 0.026	7.554	7.648	7.636
	5	0.947	0.947	0.948	0.026 / 0.027	0.028 / 0.025	0.028 / 0.024	13.369	13.447	13.466
	15	0.947	0.947	0.948	0.026 / 0.027	0.028 / 0.025	0.027 / 0.025	21.968	21.992	22.085

Table 1 reports the Monte Carlo averages of these quantities when both disturbances are Gaussian, and the predictions are calculated for $k = 1, 5$ and 15 prediction horizons. The table shows that the three procedures are very similar. The SSB procedure seems to be slightly better specially when the sample size is small and the prediction horizon increases. This result is illustrated in Figure 1 that plots kernel estimates of the ST, WS and SSB densities for the 15-steps ahead predictions for one particular series generated by each of the three models considered with $T = 50, 100$ and 500 together with the empirical density. Note that when the signal to noise ratio is small, i.e. $q = 0.1$, the SSB procedure seems to be more similar to the empirical densities than the other procedures.

Table 2, that reports the results when ε_t is $\chi^2_{(1)}$ and η_t is Gaussian, shows that the mean coverage of the ST intervals is close to the nominal. However, they are not able of dealing with the asymmetry in the distribution of ε_t. The average coverage in the left tail is smaller than in the right tail. The difference between the coverage in both tails is larger in the model with $q = 0.1$ where the signal is relatively small with respect to the noise which has a non-Gaussian distribution. Note that the lack of capability of the ST intervals to deal with the asymmetry in the distribution of ε_t is larger the larger the sample size. On the other hand, the coverages of the WS and SSB intervals are rather similar with SSB being slightly closer to the nominal, for almost all models and sample sizes considered. Both bootstrap intervals are able to cope with the asymmetry of the distribution of ε_t. Consequently, according to the results reported in Table 2, using the much simpler SSB method does not imply a worse performance of the prediction intervals. Figure 2 illustrates these results plotting the kernel density of the simulated y_{T+1} together with the ST, WS and SSB densities obtained with a particular series generated by each of the models and sample sizes considered. This figure also illustrates the lack of fit of the ST density when $q = 0.1$ and 1. On the other hand, the shapes of the WS and SSB densities are similar, with SSB being always closer to the empirical.

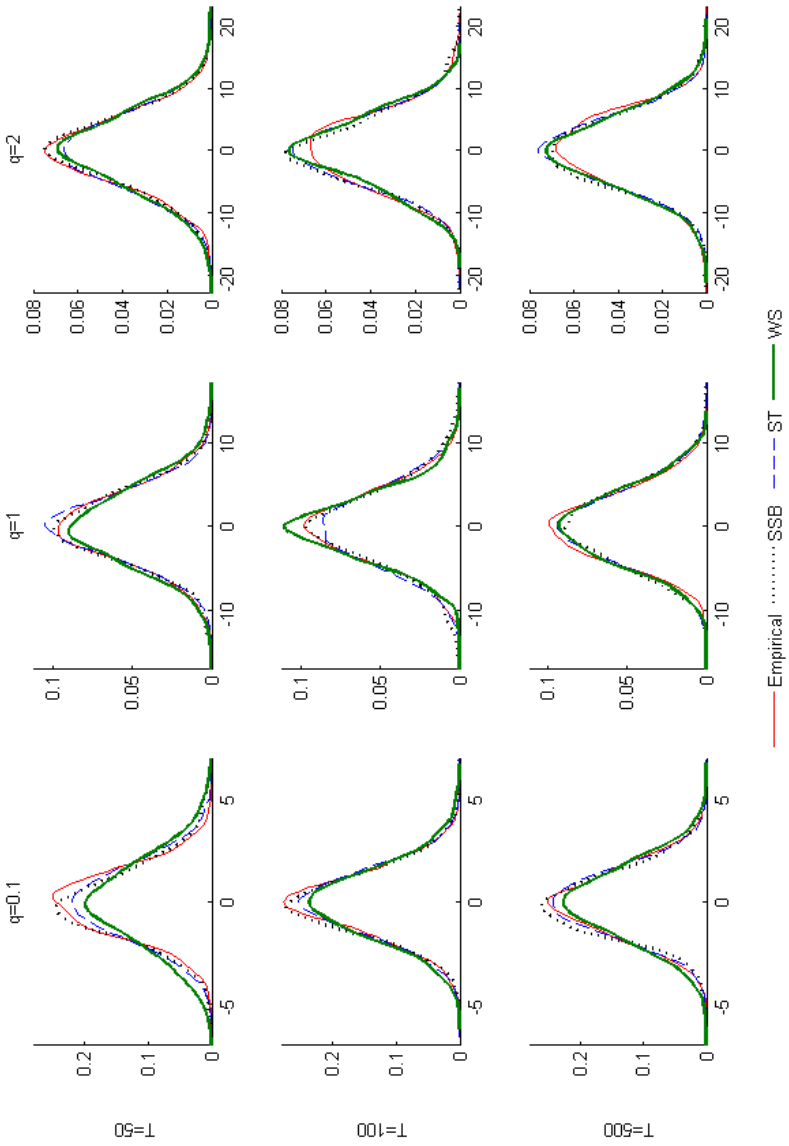


Figure 1: Kernel estimates densities of y_{T+k} for $k = 15$. Normal case.

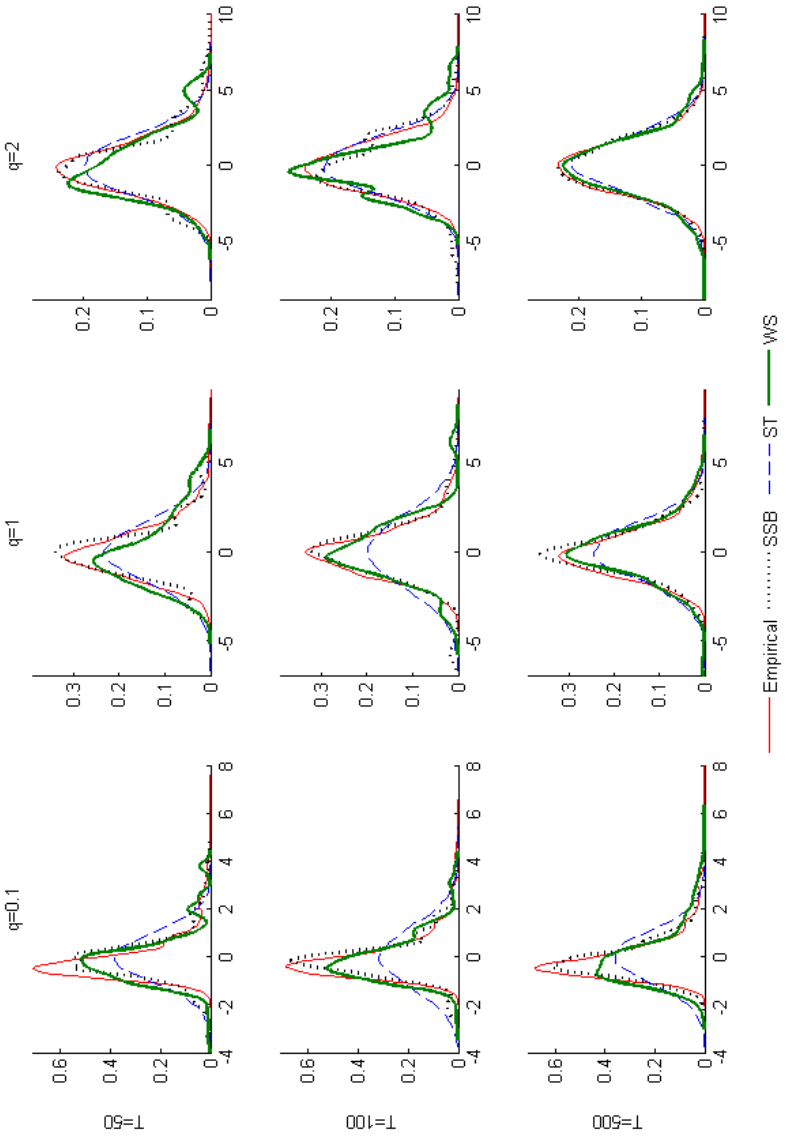


Figure 2: Kernel estimates densities of y_{T+k} for $k = 1$. $\chi^2_{(1)}$ case.

5 Application

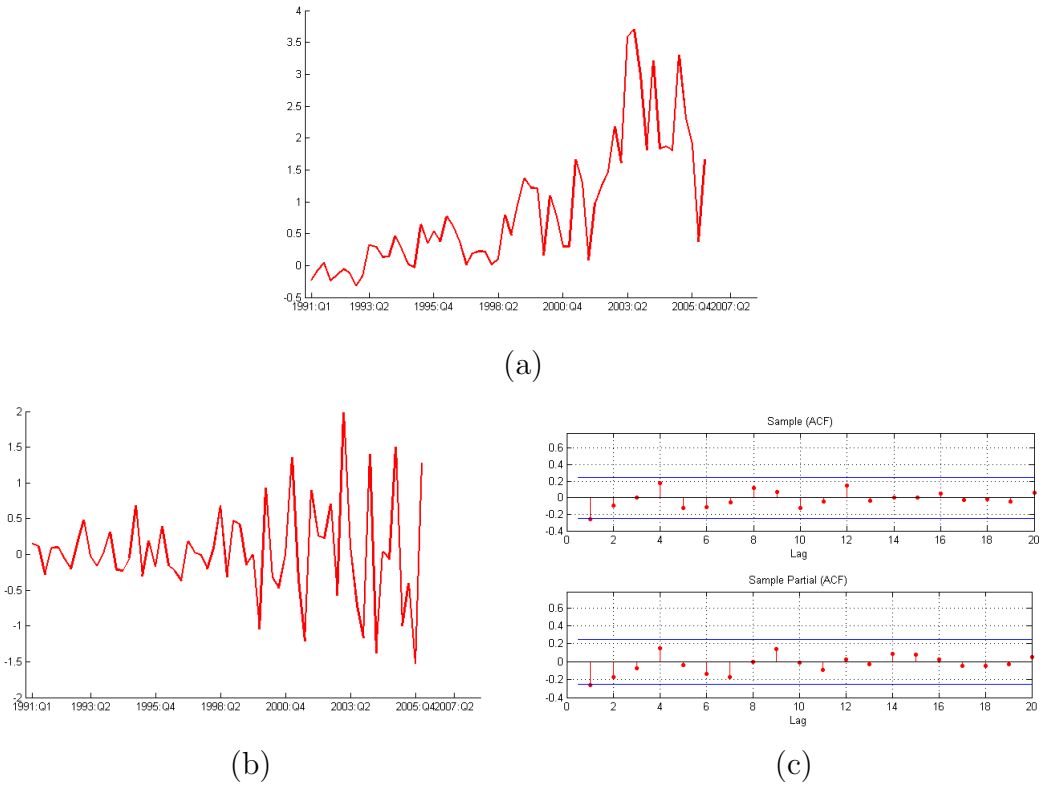
We illustrate the performance of the proposed procedure to construct bootstrap prediction intervals by implementing it on the standardized quarterly mortgages change in home equity debt outstanding, unscheduled payments, observed from 1st quarter of 1991 to the 2nd quarter of 2007 (Mortgages). The series is plotted in the panel (a) of Figure 3, which shows that it is not stationary. Its first differences are plotted in panel (b) together with its correlogram and partial correlogram, in panel (c). The pattern of the sample correlations and the partial correlations suggests that a moving average process of order one may represent adequately the dependence on the first differences of the series. Consequently, the local level model in (4) could be adequate for fitting the series of Mortgages. On the other hand, Table 3 reports several descriptive statistics for the first differences of Mortgages. This series shows excess of kurtosis and positive asymmetry with a non-Gaussian distribution reflected in small p-values for the Jarque-Bera and the Lilliefors tests for Normality.

Table 3: Descriptive statistics

Series	Δ (Mortgage) (USD billions)
Sample Size	65
Mean	0.02
Standard Dev.	0.65
Skewness	0.38
Kurtosis	4.16
Jarque-Bera (p-value)	0.008
Lillilifors (p-value)	0.004
Q(20)(p-value)	0.750

We use the observations from the 1st quarter of 1991 up to the 1st quarter of 2001, $T = 61$, for fitting the model, leaving the rest of them for evaluating the sample forecast performance of the procedure.

The QML estimates of the parameters are given by $\hat{\sigma}_\varepsilon^2 = 0.126$ and $\hat{q} = 0.671$. These estimates are used for running the Kalman filter, to obtain estimates the



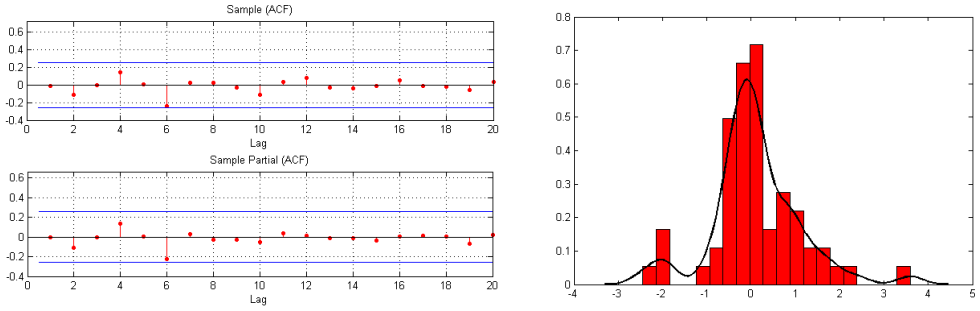
(a)

(b) (c)

Figure 3: (a) The Mortgages series. (b) First difference of Mortgages. (c) Sample Autocorrelation and Partial-Autocorrelation of the first difference of the Mortgages data.

innovations and their variances. Figure 4 plots the correlogram and a kernel estimates of the density of the within sample standardized one-step ahead errors.

The correlations and partial correlations are not any longer significant. However, the density of the errors suggests that they are obviously far from Normality. Therefore, the local level model seems appropriate to represent the dependencies in the conditional mean of the Mortgages series although for predicting future values it is convenient to implement a procedure that takes into in account the non-Normality of the errors. We construct prediction intervals up to 5 steps ahead using the ST, WS and SSB procedures. The resulting intervals are plotted in Figure 5 together with the observed values of the Mortgages series. First, observe that the two bootstrap procedures generate very similar intervals which are wider



(a) (b)

Figure 4: (a) Sample Autocorrelation and Partial-Autocorrelation of standardized one-step ahead error. (b) Empirical density and histogram for the standardized one-step ahead error.

than the ST intervals, as expected given that they incorporate the uncertainty due to parameter estimation. For two prediction horizons, the observations corresponding to the 2nd quarter of 2006 and the 1st quarter of 2007, fall outside the ST prediction interval. However, both bootstrap procedures still contain these two values. It is important to note that although bootstrap procedures are computational intensive, in this application with $B = 2000$ bootstrap replicates, the BSS procedure requires 110 seconds using a MATLAB algorithm in an AMD Athlon 2.00GHz processor of a PC desktop with 2.00Gb of RAM. However, the Wall and Stoffer (2002) bootstrap procedure requires 160 seconds. There is a reduction of 31% in the computer time required.

6 Conclusion

This paper proposes a new procedure to obtain bootstrap prediction intervals in the context of State Space models. Bootstrap intervals are of great interest when predicting future values of a series of interest as they are able to incorporate parameter uncertainty and do not rely on any particular assumption on the error distribution. Wall and Stoffer (2002) propose a bootstrap procedure to obtain

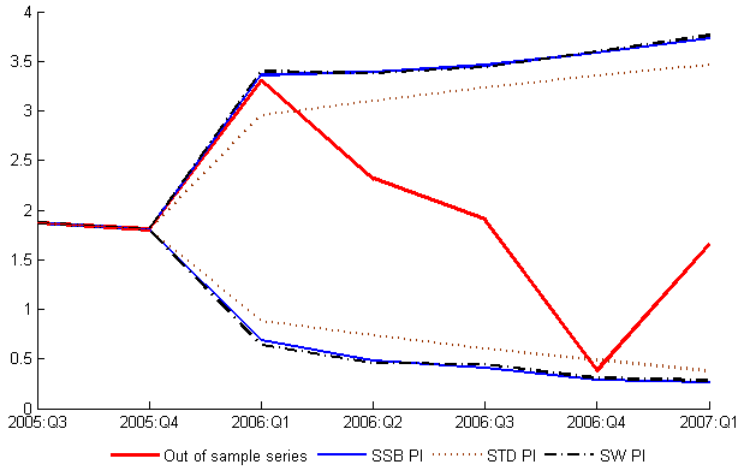


Figure 5: Bootstrap and standard prediction intervals for the out of sample forecasting evaluation for Mortgage series.

the density of the prediction errors in two steps. First, the uncertainty due to parameter estimation is taken into in account and then the uncertainty due to the distribution of the prediction error is considered. Furthermore, their procedure is implemented using the backward representation of the model in order to keep fixed the last observations of the series when bootstrapping the parameter estimates.

The procedure proposed in this paper has three advantages. First, it is based on obtaining directly the density of future observations instead of the density of the errors. Furthermore, this density is obtained in one single step that incorporate simultaneously the uncertainty due to the parameters estimation and the uncertainty due to the error distribution. Finally and more important, the bootstrap procedure proposed in this paper does not rely on the backward representation. As a consequence, our procedure is much simpler from a computational point of view and can be extended to models without a backward representation.

We analyze the small sample behavior of the proposed bootstrap intervals and compare it with those of the intervals proposed by Wall and Stoffer (2002) and the intervals based on assuming known parameters and a Normal distribution of the errors. We show that our procedure, although much simpler, has slightly better properties than the bootstrap intervals of Wall and Stoffer (2002). As expected,

we also show that bootstrap intervals are more adequate than standard intervals mainly in the presence of non-Normal errors. In general, the standard intervals are thinner than expected to have the nominal coverage and cannot deal with asymmetries.

Finally, our proposed bootstrap procedure to obtain prediction intervals in State Space models is illustrated by implementing it to obtain intervals for future values of a series of Mortgages modelled by the local level model. We show that there is an important improvement in terms of computer time when implementing our proposed procedure with respect to implementing the procedure proposed by Wall and Stoffer (2002).

When fitting State Space models to represent the dynamic evolution of a time series, it is often of interest to obtain prediction not only of future values of the series but also of future values of the unobserved states. We are also working on the adequacy of the proposed bootstrap prediction intervals when implemented with this goal. A issue left for further research is the implementation of the proposed procedure when the system of matrices are time-varying.

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Notes

[1]We try alternative methods as the bias-corrected and the acceleration bias-corrected with similar results; see Efron (1987) for a definition of these intervals.

[2]We are particularly interested in dealing with this distribution due to its relation with the linear transformation of the Autoregressive Stochastic Volatility Model; see, for instance, Harvey et al. (1994). Results for other distributions are similar and are not reported to save space. Available for the authors upon request.

[3]All, simulation, estimation and prediction has been done with programs developed by the authors using the software MATLAB, version 7.2.